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The Principal Spectrum for Linear Nonautonomous Parabolic PDEs of Second Order: Basic Properties[1]

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order

$$u_t = \sum_{i,\,j=1}^{N} a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(t, x) \frac{\partial u}{\partial x_i} + a_0(t, x)\, u, \qquad x \in \Omega,$$

on a bounded domain $\Omega \subset \mathbb{R}^N$, with Dirichlet or Robin boundary conditions. A canonically defined one-dimensional subbundle $\mathscr{S}$ (corresponding to the solutions that are globally defined and of the same sign) serves as an analog of principal eigenfunction. The principal spectrum is defined to be the dynamical (Sacker–Sell) spectrum of the linear skew-product flow on $\mathscr{S}$. Characterizations of principal spectrum in terms of (logarithmic) growth rates of positive solutions are given. Finally, monotonicity of the principal spectrum with respect to zero order terms is proved. © 2000 Academic Press

It is a standard and well-known fact that for a linear parabolic partial differential equation (PDE) of second order

$$u_t = \Delta u + a_0(x)\, u, \qquad t > 0,\ x \in \Omega$$

complemented with homogeneous Dirichlet or Neumann boundary conditions, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary and the coefficient a_0 is sufficiently regular, the eigenvalue $\lambda_{\max}$ for the eigenvalue problem

$$(\Delta + a_0(\cdot))\, u + \lambda u = 0 \quad \text{(plus the boundary conditions)}$$

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having the largest real part (the *principal eigenvalue*) is real, simple, and an eigenfunction corresponding to it (a *principal eigenfunction*) v can be chosen so that $v(x) > 0$ for all $x \in \Omega$. Here the Laplacian Δ can be replaced by an arbitrary uniformly elliptic second order differential operator with sufficiently regular coefficients.

The concepts of principal eigenvalue and principal eigenfunction were extended by P. Hess and some of his students to the case where the coefficients of the equation are allowed to depend periodically on time, see P. Hess' monograph [10].

The aim of the present paper is to initiate a theory extending the above notions to the case of coefficients depending on time in a not necessarily periodic (nor even almost periodic) way.

Our starting point is a linear nonautonomous partial differential equation of second order

$$u_t = \sum_{i,j=1}^{N} a_{ij}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(t,x) \frac{\partial u}{\partial x_i} + a_0(t,x)\, u, \qquad t \in \mathbb{T},\ x \in \Omega, \tag{E}$$

where $\mathbb{T} = (-\infty, \infty)$ or $\mathbb{T} = [0, \infty)$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with boundary $\partial\Omega$ of class $C^{1+\alpha}$, $\alpha > 0$.

Equation (E) is considered either with homogeneous Dirichlet boundary conditions

$$u(t,x) = 0, \qquad t > 0,\ x \in \partial\Omega, \tag{DBC}$$

or with regular oblique boundary conditions (Robin boundary conditions)

$$\frac{\partial u}{\partial \beta}(t,x) + c(x)\, u(t,x) = 0, \qquad t > 0, \quad x \in \partial\Omega, \tag{RBC}$$

where $\beta \in C^1(\partial\Omega, \mathbb{R}^N)$ is a nontangential vector field pointing out of Ω, and $c \in C^1(\partial\Omega)$ is a nonnegative function.

We present now the main ideas of the construction. We look upon all the time translates of the coefficients of the original equation as elements of some compact metric space B. The topology on that space of parameters should be strong enough to allow continuous dependence of solutions on parameters. On the other hand, that topology should be sufficiently weak to have the continuity of the time translation. We note that one can conceive of many (notably different) theories guaranteeing such existence and continuity results (see Henry [9], or Amann [2]). We have also in mind possible extensions of our results to cover stochastic perturbations. These are the reasons why we have chosen to extract all the needed properties of the theories of existence and continuous dependence in the form of eleven axioms (Section 1). At the end of that section (Subsection 1.1) we give two examples of such theories.

In Section 2 we define the main object of study in this paper. All the equations whose coefficients belong to the closure of the family of time translates of the original equation (the *hull*) generate a linear skew-product semiflow on a Banach bundle. By a result due to P. Poláčik and I. Tereščák, and, independently, to the author (Theorem 2.3), there is a canonically defined one-dimensional invariant subbundle (the *Kreĭn–Rutman bundle*), corresponding to globally positive solutions. This object is an analog of the principal eigenfunction. Similarly, the dynamical (Sacker–Sell) spectrum of the restriction of the semiflow to the Kreĭn–Rutman bundle serves as an analog of the principal eigenvalue (and is called the *principal spectrum*). In the last two subsections of Section 2 we give characterizations of the principal spectrum in terms of the original equation only, i.e. without considering the equations from the hull.

The last section deals with the monotonicity of the principal spectrum with respect to the zero order coefficients.

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1. ASSUMPTIONS ON THE COEFFICIENTS

Denote by $\mathcal{M}(\mathbb{T}\times\Omega, \mathbb{R}^m)$ the set of all Lebesgue measurable functions from $\mathbb{T}\times\Omega$ into $\mathbb{R}^m$.

For $f\in\mathcal{M}(\mathbb{T}\times\Omega, \mathbb{R}^m)$ and $t\in\mathbb{T}$ we denote the *t-translate* of f by $f\cdot t$,

$$(f\cdot t)(s,x) := f(t+s,x) \qquad \text{for a.e.} \quad s\in\mathbb{T},\, x\in\Omega.$$

The function $f\cdot t$ belongs to $\mathcal{M}(\mathbb{T}\times\Omega, \mathbb{R}^m)$, too.

We put $\mathcal{M} := \mathcal{M}(\mathbb{T}\times\Omega, \mathbb{R}^{N^2+N+1})$.

The first set of axioms concerns the structure of the space B of allowable coefficients.

(A1) B is a convex compact metrizable subset of a topological vector space, contained (set-theoretically) in $\mathcal{M}$.

(A2) If $(b_{ij}, b_i, b_0)\in B$ then $(b_{ij}, b_i, b_0)\cdot t\in B$ for all $t\in\mathbb{T}$.

(A3) The mapping $B\times\mathbb{T}\ni((b_{ij}, b_i, b_0), t)\mapsto(b_{ij}, b_i, b_0)\cdot t\in B$ is continuous.

We will refer to the closure $\mathcal{H}(b_{ij}, b_i, b_0)$ in B of the set $\{(b_{ij}, b_i, b_0)\cdot t : t\in\mathbb{T}\}$ as the *hull* of $(b_{ij}, b_i, b_0)\in B$.

The next set of axioms postulates the existence and continuous dependence of solutions of the boundary-initial value problem on initial data and parameters.

(A4) To each $(b_{ij}, b_i, b_0) \in B$, $1 < p < \infty$ and $u_0 \in L^p(\Omega)$ one assigns a continuous function $u_p(\,\cdot\,; (b_{ij}, b_i, b_0), u_0)$ defined on $[0, \infty)$ and taking values in $L^p(\Omega)$, such that $u_p(0; (b_{ij}, b_i, b_0), u_0) = u_0$ and

$$u_p(\,\cdot\,; (b_{ij}, b_i, b_0) \cdot s, u_p(s; (b_{ij}, b_i, b_0), u_0)) = u_p(\,\cdot + s; (b_{ij}, b_i, b_0), u_0) \tag{1.1}$$

for each $(b_{ij}, b_i, b_0) \in B$, $1 < p < \infty$, $u_0 \in L^p(\Omega)$ and $s \in [0, \infty)$.

We will refer to $u_p(\,\cdot\,; (b_{ij}, b_i, b_0), u_0)$ as the *solution* of equation

$$u_t = \sum_{i,\, j=1}^{N} b_{ij}(t, x)\, u_{x_j x_i} + \sum_{i=1}^{N} b_i(t, x)\, u_{x_i} + b_0(t, x)\, u, \quad t > 0,\ x \in \Omega, \quad (\mathrm{E}_{(b_{ij},\, b_i,\, b_0)})$$

satisfying the boundary condition

$$u(t, x) = 0 \quad [\text{or } (\partial u/\partial \beta)(t, x) + c(x)\, u(t, x) = 0], \qquad t > 0, \quad x \in \partial\Omega,$$

and the initial condition

$$u(0, x) = u_0(x).$$

It should be emphasized here that we do not assume that the solution is classical.

(A5) For each $1 < p < \infty$ and $T \geqslant 0$ the mapping

$$L^p(\Omega) \times B \ni (u_0, (b_{ij}, b_i, b_0)) \mapsto u_p(\,\cdot\,; (b_{ij}, b_i, b_0), u_0) \in C([0, T], L^p(\Omega))$$

is continuous.

Next we deal with the smoothing properties of the solution operator.

(A6) For each $(b_{ij}, b_i, b_0) \in B$, $1 < p < \infty$, $u_0 \in L^p(\Omega)$ and $t > 0$ one has $u_p(t; (b_{ij}, b_i, b_0), u_0) \in C^1(\bar{\Omega})$.

The above axiom allows us to suppress the subscript p and write simply $u(t; (b_{ij}, b_i, b_0), u_0)$ (for $t > 0$).

(A7) For each $(b_{ij}, b_i, b_0) \in B$, $1 < p < \infty$ and $t > 0$ the linear operator

$$L^p(\Omega) \ni u_0 \mapsto u(t; (b_{ij}, b_i, b_0), u_0) \in C^1(\bar{\Omega})$$

is completely continuous.

(A8) For each $(b_{ij}, b_i, b_0) \in B$, $1 < p < \infty$ and $t > 0$ the linear operator

$$[u_0 \mapsto u(t; (b_{ij}, b_i, b_0), u_0)] \in \mathscr{L}(L^p(\Omega), C^1(\bar{\Omega}))$$

extends to an operator in $\mathscr{L}(C^1(\bar{\Omega})^*, C^1(\bar{\Omega}))$.

Except at $t = 0$ we have also a stronger form of continuity:

(A9) For each $1 < p < \infty$ and $0 < T_1 \leqslant T_2$ the mapping

$$\begin{aligned} B \ni (b_{ij}, b_i, b_0) \mapsto [u_0 \mapsto u(\,\cdot\,; (b_{ij}, b_i, b_0), u_0)] \\ \in C([T_1, T_2], \mathscr{L}(C^1(\bar{\Omega})^*, C^1(\bar{\Omega}))) \end{aligned}$$

is continuous.

Let ν stand for the normalized normal vector field on $\partial\Omega$ pointing out of Ω.

We denote by $\mathbf{e}$ the principal eigenfunction of the Laplacian on Ω with boundary conditions (DBC) or (RBC), normalized so that $\|\mathbf{e}\|_{L^2(\Omega)} = 1$.

Each of the spaces $C(\bar{\Omega})$ or $C^1(\bar{\Omega})$ is *ordered* by the (standard) *cone* $C(\bar{\Omega})^+ := \{u \in C(\bar{\Omega}) : u(x) \geqslant 0 \text{ for all } x \in \bar{\Omega}\}$ (resp. $C^1(\bar{\Omega})^+ := \{u \in C^1(\bar{\Omega}) : u(x) \geqslant 0 \text{ for all } x \in \bar{\Omega}\}$). Similarly, for $1 \leqslant p \leqslant \infty$ the Banach space $L^p(\Omega)$ is ordered by the (standard) cone $L^p(\Omega)^+ := \{u \in L^p(\Omega) : u(x) \geqslant 0 \text{ for a.e. } x \in \Omega\}$.

If the cone C^+ has nonempty interior C^{++}, we say that (C, C^+) is a *strongly ordered* Banach space. Both the spaces $(C(\bar{\Omega}), C(\bar{\Omega})^+)$ and $(C^1(\bar{\Omega}), C^1(\bar{\Omega})^+)$ are strongly ordered: $C(\bar{\Omega})^{++} = \{u \in C(\bar{\Omega}) : u(x) > 0 \text{ for all } x \in \bar{\Omega}\}$, $C^1(\bar{\Omega})^{++} = \{u \in C^1(\bar{\Omega}) : u(x) > 0 \text{ for all } x \in \Omega\}$. On the other hand, the spaces $(L^p(\Omega), L^p(\Omega)^+)$, $1 \leqslant p < \infty$, are not strongly ordered.

By $C^1_0(\bar{\Omega})$ we denote the Banach space $\{u \in C^1(\bar{\Omega}) : u(x) = 0 \text{ for all } x \in \partial\Omega\}$, with the C^1 norm. The space $C^1_0(\bar{\Omega})$ is ordered by the cone $C^1_0(\bar{\Omega})^+ = \{u \in C^1_0(\bar{\Omega}) : u(x) \geqslant 0 \text{ for all } x \in \bar{\Omega}\}$ with nonempty interior $C^1_0(\bar{\Omega})^{++} = \{u \in C^1_0(\bar{\Omega}) : u(x) > 0 \text{ for all } x \in \Omega \text{ and } (\partial u/\partial \nu)(x) < 0 \text{ for all } x \in \partial\Omega\}$. We write $C^1_{\mathscr{B}}(\bar{\Omega})$ for $C^1_0(\bar{\Omega})$ in case of Dirichlet boundary conditions, and for $C^1(\bar{\Omega})$ in case of Robin boundary conditions.

For u, v in an ordered Banach space (C, C^+), $u \leqslant_C v$ means $v - u \in C^+$, and $u <_C v$ means $u \leqslant_C v$ and $u \neq v$. If C is strongly ordered, $u \ll_C v$ means $v - u \in C^{++}$.

A cone C^+ in the Banach space C is called *normal* if there is a constant $K > 0$ such that for each $u, v \in C$ the inequality $0 <_C u <_C v$ implies $\|u\|_C \leqslant K \|v\|_C$, where $\|\cdot\|_C$ stands for the norm in C. An ordered Banach space (C, C^+) with C^+ normal is referred to as a *normally ordered* Banach space. It is well known (see e.g. Amann [1]) that a normally ordered

Banach space can be (equivalently) renormed in such a way that the constant K in the definition of normality equals 1 (such a norm is called *monotone*).

The ordered Banach spaces $(C(\bar{\Omega}), C(\bar{\Omega})^+)$ and $(L^p(\Omega), L^p(\Omega)^+)$, $1 \leqslant p \leqslant \infty$, are normally ordered. Their respective norms are monotone. On the other hand, $(C^1_{\mathscr{B}}(\bar{\Omega}), C^1_{\mathscr{B}}(\bar{\Omega})^+)$ is not normally ordered.

For $u \in C(\bar{\Omega})$ denote

$$\|u\|_{C(\bar{\Omega}), \mathbf{e}} := \inf\{\alpha \geqslant 0 : -\alpha \mathbf{e}(x) \leqslant u(x) \leqslant \alpha \mathbf{e}(x) \text{ for all } x \in \bar{\Omega}\}.$$

Let $C(\bar{\Omega})_{\mathbf{e}}$ stand for the set of all those $u \in C(\bar{\Omega})$ for which $\|u\|_{C(\bar{\Omega}), \mathbf{e}} < \infty$. $C(\bar{\Omega})_{\mathbf{e}}$ is a Banach space, normally ordered by the cone $C(\bar{\Omega})^+_{\mathbf{e}} := \{u \in C(\bar{\Omega})_{\mathbf{e}} : u(x) > 0 \text{ for all } x \in \Omega\}$. The interior $C(\bar{\Omega})^{++}_{\mathbf{e}}$ of $C(\bar{\Omega})^+_{\mathbf{e}}$ is nonempty: $C(\bar{\Omega})^{++}_{\mathbf{e}} = \{u \in C(\bar{\Omega})_{\mathbf{e}}$: there are $0 < \alpha < \beta$ such that $\alpha \mathbf{e}(x) \leqslant u(x) \leqslant \beta \mathbf{e}(x)$ for all $x \in \bar{\Omega}\}$. In case of Robin boundary conditions (RBC), $C(\bar{\Omega})_{\mathbf{e}}$ equals $C(\bar{\Omega})$ (up to renorming).

Similarly, for $u \in L^2(\Omega)$ denote

$$\|u\|_{L^2(\Omega), \mathbf{e}} := \inf\{\alpha \geqslant 0 : -\alpha \mathbf{e}(x) \leqslant u(x) \leqslant \alpha \mathbf{e}(x) \text{ for a.e. } x \in \Omega\}.$$

Let $L^2(\Omega)_{\mathbf{e}}$ stand for the set of all those $u \in L^2(\Omega)$ for which $\|u\|_{L^2(\Omega), \mathbf{e}} < \infty$. $L^2(\Omega)_{\mathbf{e}}$ is a Banach space, normally ordered by the cone $L^2(\Omega)^+_{\mathbf{e}} := \{u \in L^2(\Omega)_{\mathbf{e}} : u(x) \geqslant 0 \text{ for a.e. } x \in \Omega\}$, with nonempty interior $L^2(\Omega)^{++}_{\mathbf{e}} = \{u \in L^2(\Omega)_{\mathbf{e}}$: there are $0 < \alpha < \beta$ such that $\alpha \mathbf{e}(x) \leqslant u(x) \leqslant \beta \mathbf{e}(x)$ for a.e. $x \in \Omega\}$. In case of Robin boundary conditions, $L^2(\Omega)_{\mathbf{e}}$ equals $L^\infty(\Omega)$ (up to renorming).

Instead of writing $\|\cdot\|_{C(\bar{\Omega}), \mathbf{e}}$ or $\|\cdot\|_{L^2(\Omega), \mathbf{e}}$ we will write $\|\cdot\|_{\mathbf{e}}$. Similarly, instead of $\leqslant_{C(\bar{\Omega})_{\mathbf{e}}}$ or $\leqslant_{L^2(\Omega)_{\mathbf{e}}}$ we will write $\leqslant_{\mathbf{e}}$, etc. This should not cause misunderstandings.

The next axioms concern the monotone dependence of the solution on initial data and zero order parameters.

(A10) For each $(b_{ij}, b_i, b_0) \in B$, $1 < p < \infty$, $u_0 \in L^p(\Omega)^+ \setminus \{0\}$ and $t > 0$ one has $u(t; (b_{ij}, b_i, b_0), u_0) \in C^1_{\mathscr{B}}(\bar{\Omega})^{++}$.

(A11) Assume that $b_{0,1}(t, x) \leqslant b_{0,2}(t, x)$ for a.e. $(t, x) \in \mathbb{T} \times \Omega$. Then for each $1 < p < \infty$, $u_{0,1}, u_{0,2} \in L^p(\Omega)$, $u_{0,1} \leqslant_{L^p(\Omega)} u_{0,2}$ we have

$$u(t; (b_{ij}, b_i, b_{0,1}), u_{0,1}) \leqslant_{\mathbf{e}} u(t; (b_{ij}, b_i, b_{0,2}), u_{0,2})$$

for all $t > 0$.

1.1. *Examples*

In the present subsection we briefly outline two important situations where our Axioms (A1)–(A11) hold.

(I) *The classical case.* Consider the equation

$$u_t = \Delta u + a_0(t, x)\, u, \qquad t \in \mathbb{T}, \quad x \in \Omega, \tag{1.2}$$

with the homogeneous Dirichlet or Robin boundary conditions, where $a_0 : \mathbb{T} \times \bar{\Omega} \to \mathbb{R}$ is a bounded continuous function, uniformly Hölder in x with exponent α, $0 < \alpha \leqslant 1$, and uniformly Hölder in t with exponent $\alpha/2$.

Put $\| \cdot \|_{\mathbf{b}}$ to be the supremum norm, $h_x\{\cdot\}$ to be the Hölder norm in x (with exponent α) and $h_x\{\cdot\}$ to be the Hölder norm in t (with exponent $\alpha/2$). We define B as the set of those continuous functions (b_{ij}, b_i, b_0) from $\mathbb{T} \times \bar{\Omega}$ into $\mathbb{R}^{N^2+N+1}$ such that $b_{ij} \equiv \delta_{ij} 1$, $i, j = 1,..., N$, where δ_{ij} stands for the Kronecker delta, $b_i \equiv 0$ for $i = 1,..., N$, $\|b_0\|_{\mathbf{b}} \leqslant \|a_0\|_{\mathbf{b}}$, $h_x\{b_0\} \leqslant h_x\{a_0\}$ and $h_t\{b_0\} \leqslant h_t\{a_0\}$. The set B is endowed with the topology of uniform convergence on compact sets.

The set B is clearly convex. Also, the closedness of B under translations by $t \in \mathbb{T}$ is straightforward. A standard application of the Ascoli–Arzelà theorem yields that B is a compact metrizable space. The continuity of the mapping

$$B \times \mathbb{T} \ni ((b_{ij}, b_i, b_0), t) \mapsto (b_{ij}, b_i, b_0) \cdot t \in B$$

follows by the equicontinuity of the elements of B on compact subsets. Consequently, Axioms (A1)–(A3) are satisfied.

The continuity and smoothness properties of the solution as described in Axioms (A4)–(A7) are standard (see e.g. Henry's book [9]).

It should be remarked that the solution obtained is *classical*: denoting by $u(t, x)$ the evaluation of $u(t)$ at x, we have that the derivatives u_t and $u_{x_i x_j}$ exist and are continuous in $(0, \infty) \times \Omega$, and the equation is satisfied there pointwise.

Axioms (A8)–(A9) are consequences of the existence of a Green's function (see Friedman [7]).

Finally, Axioms (A10)–(A11) follow from the parabolic strong maximum and the Hopf maximum principles.

In the case where $\mathbb{T} = (-\infty, \infty)$ and a_0 is almost periodic in t uniformly in $x \in \bar{\Omega}$ one can take as B an appropriate subset of some Banach space of bounded continuous functions defined on $\mathbb{R}$, with the uniform topology (for details see e.g. Chapters 1 and 2 in Fink [6]). It should be mentioned here that "dynamical" theory of nonautonomous parabolic PDEs (even nonlinear ones) is the most developed in the almost periodic case for $N = 1$ (see e.g. Shen and Yi [19]). One should also mention a paper [3] by Bernfeld, Hu and Vuillermot, and some of Vuillermot's earlier papers quoted there.

(II) *The L^∞ case.* Consider the one-dimensional equation

$$u_t = u_{xx} + a_0(t, x)\, u, \qquad t \in \mathbb{T}, \quad x \in (0, \pi), \tag{1.3}$$

with the Dirichlet boundary conditions, where $a_0 : \mathbb{T} \times \bar{\Omega} \to \mathbb{R}$ belongs to $L^\infty(\mathbb{T} \times (0, \pi))$.

B is defined as the set of those $(b_{ij}, b_i, b_0) \in L^\infty(\mathbb{T} \times (0, \pi), \mathbb{R}^3)$ such that $b_{11} \equiv 1$, $b_1 \equiv 0$, and the essential supremum of b_0 does not exceed the essential supremum of a_0. The set B is considered with the weak-* topology.

The fulfillment of Axioms (A1)–(A7) was proved in the paper [5] by S.-N. Chow, K. Lu and J. Mallet-Paret. The fact that Axioms (A8)–(A9) are satisfied can be proved via the use of a Green's function by passing to the adjoint equation, along the lines of the author's paper [14]. Axioms (A10)–(A11) follow again from the maximum principles.

2. DEFINITION OF THE PRINCIPAL SPECTRUM

In the present section we fix $(a_{ij}, a_i, a_0) \in B$.

The principal spectrum of equation (E) will be defined as the dynamical spectrum of some linear skew-product dynamical system on a canonically defined one-dimensional vector bundle.

The starting point will be construction of a linear skew-product semidynamical system on a product Banach bundle $\mathscr{B}$.

2.1. *Construction of the Bundle $\mathscr{B}$*

The construction depends on whether $\mathbb{T} = (-\infty, \infty)$ or $\mathbb{T} = [0, \infty)$.

Case 1 ($\mathbb{T} = (-\infty, \infty)$). We put $\mathbb{A} := \mathscr{H}(a_{ij}, a_i, a_0)$. Generic elements in $\mathbb{A}$ will be denoted by $\mathbf{b} := (b_{ij}, b_i, b_0)$, $\mathbf{c} := (c_{ij}, c_i, c_0)$, etc.

For each $\mathbf{b} \in \mathbb{A}$ and $t \geqslant 0$ we define

$$\phi_t(\mathbf{b})\, u_0 := (\mathbf{b} \cdot t, u(t; (b_{ij}, b_i, b_0), u_0)), \qquad u_0 \in L^2(\Omega).$$

Case 2 ($\mathbb{T} = [0, \infty)$). For a function $f \in \mathscr{M}(\mathbb{R} \times \Omega, \mathbb{R}^m)$ and $t \in \mathbb{R}$ by $f|_t$ we denote $f \cdot t$ restricted to $[0, \infty) \times \Omega$.

Let $\bar{\delta}$ stand for the metric on $\mathscr{H}(a_{ij}, a_i, a_0)$. A function $\mathbf{b} \in \mathscr{M}(\mathbb{R} \times \Omega, \mathbb{R}^{N^2+N+1})$ belongs to $\mathbb{A}$ if there is a sequence $t_n \to \infty$ as $n \to \infty$, such that for each $t \in \mathbb{R}$ one has $\bar{\delta}((a_{ij}, a_i, a_0) \cdot (t + t_n), \mathbf{b}|_t) \to 0$ as $n \to \infty$.

We define a metric δ on $\mathbb{A}$ by the formula

$$\delta(\mathbf{b}, \mathbf{c}) := \sum_{k=0}^{\infty} 2^{-k} \bar{\delta}(\mathbf{b}|_{-k}, \mathbf{c}|_{-k}).$$

LEMMA 2.1. *$(\mathbb{A}, \delta)$ is a nonempty compact metric space.*

Proof. The result follows by a standard application of the diagonal process. ∎

The following result is standard.

LEMMA 2.2. *The mapping*

$$\mathbb{R} \times \mathbb{A} \ni (t, \mathbf{b}) \mapsto \mathbf{b} \cdot t \in \mathbb{A}$$

is continuous.

For each $\mathbf{b} \in \mathbb{A}$ and $t > 0$ we define

$$\phi_t(\mathbf{b})\, u_0 := (\mathbf{b} \cdot t, u(t; \mathbf{b}|_0, u_0)), \qquad u_0 \in L^2(\Omega).$$

In the sequel we will say for a sequence $t_n \to \infty$ (slightly abusing the language) "$(a_{ij}, a_i, a_0) \cdot t_n \to \mathbf{b}$ as $n \to \infty$" instead of saying "for each $t \in \mathbb{R}$, $\bar{\delta}((a_{ij}, a_i, a_0) \cdot (t + t_n), \mathbf{b}|_t) \to 0$ as $n \to \infty$."

2.2. *Kreĭn–Rutman Bundle and Principal Spectrum*

We define a linear skew-product semidynamical system $\Phi = \{\Phi_t\}_{t \geqslant 0}$ on the Banach bundle $\mathscr{B} = \mathbb{A} \times L^2(\Omega)$ by

$$\Phi_t(\mathbf{b}, u) := (\mathbf{b} \cdot t, \phi_t(\mathbf{b})\, u) \qquad t \geqslant 0, \mathbf{b} \in \mathbb{A}, u \in L^2(\Omega).$$

The continuity of the assignment

$$[0, \infty) \times \mathbb{A} \times L^2(\Omega) \ni (t, \mathbf{b}, u) \mapsto \Phi_t(\mathbf{b}, u) \in \mathbb{A} \times L^2(\Omega)$$

is a consequence of Axiom (A5).

Equality (1.1) yields the *cocycle equality*,

$$\phi_{t+s}(\mathbf{b}) = \phi_t(\mathbf{b} \cdot s)\, \phi_s(\mathbf{b}) \qquad s, t \geqslant 0, \mathbf{b} \in \mathbb{A}, \tag{2.1}$$

which is equivalent to the semigroup property $\Phi_{t+s} = \Phi_t \circ \Phi_s$.

By $\mathscr{Z}$ we denote the null section of $\mathscr{B}$, $\mathscr{Z} = \{(\mathbf{b}, 0) : \mathbf{b} \in \mathbb{A}\}$. The symbol $\|\cdot\|_2$ will from now on stand for the $L^2(\Omega)$-norm.

The following theorem is based on a result proved independently by the author [12] and by Poláčik and Tereščák [17] (for the details of the proof, compare the proof of Thm. 2.2 in Mierczyński [13], with $L^1(\Omega)$ replaced by $L^2(\Omega)$).

THEOREM 2.3. *There exists an invariant decomposition $\mathscr{B} = \mathscr{S} \oplus \mathscr{T}$, $\dim \mathscr{S} = 1$, having the following properties*:

(KR1) $\mathscr{S}\backslash\mathscr{Z} \subset \mathbb{A} \times (C(\bar{\Omega})_{\mathbf{e}}^{++} \cup (-C(\bar{\Omega})_{\mathbf{e}}^{++}))$.

(KR2) $\mathscr{T} \cap (\mathbb{A} \times L^2(\Omega)^+) = \mathscr{Z}$.

(KR3) *There are constants* $0 < d \leqslant 1$ *and* $\mu > 0$ *such that*

$$\frac{\|\phi_t(\mathbf{b})\, u\|_2}{\|\phi_t(\mathbf{b})\, v\|_2} \geqslant de^{\mu t} \frac{\|u\|_2}{\|v\|_2} \tag{2.2}$$

for each $\mathbf{b} \in \mathbb{A}$, $(\mathbf{b}, u) \in \mathscr{S}$, $(\mathbf{b}, v) \in \mathscr{T}$ *and* $t > 0$.

We will refer to $\mathscr{S}$ as the *Kreĭn–Rutman bundle* (KR bundle, for short). The property (KR3) is called *exponential separation.*

By $\mathscr{S}_{\mathbf{b}}$ we understand the fiber of $\mathscr{S}$ over $\mathbf{b} \in \mathbb{A}$, $\mathscr{S}_{\mathbf{b}} = \{u \in C(\bar{\Omega})_{\mathbf{e}} : (\mathbf{b}, u) \in \mathscr{S}\}$. The symbol $\mathscr{T}_{\mathbf{b}}$ is defined in an analogous way. For $\mathbf{b} \in \mathbb{A}$ by $v_{\mathbf{b}}$ we will denote the unique element of $\mathscr{S}_{\mathbf{b}} \cap C(\bar{\Omega})_{\mathbf{e}}{}^{++}$ with $\|v_{\mathbf{b}}\| = 1$.

We should pause for a while to explain what we understand under the term invariant. As our semidynamical system is defined originally for positive times, this means that if u belongs to the fiber $\mathscr{S}_{\mathbf{b}}$ then $\phi_t(\mathbf{b})\, u$ belongs to $\mathscr{S}_{\mathbf{b}\cdot t}$, for all $t \geqslant 0$. However, from (KR3) it follows that $\phi_t(\mathbf{b})$ restricted to $\mathscr{S}_{\mathbf{b}}$ is a bundle automorphism, hence for each $t > 0$ and $u \in \mathscr{S}_{\mathbf{b}}$ we can define $\phi_{-t}(\mathbf{b})\, u$ as the unique element of $\mathscr{S}_{\mathbf{b}\cdot(-t)}$ taken to u by $\phi_t(\mathbf{b}\cdot(-t))$.

Instead of writing $(\mathrm{E}_{\mathbf{b}_0})$ we will write simply $(\mathrm{E}_{\mathbf{b}})$. We say that a solution u of $(\mathrm{E}_{\mathbf{b}})$, $\mathbf{b} \in \mathbb{A}$, is *global* if it is defined on the whole of $(-\infty, \infty)$. A global solution u is called *globally positive* if $u(t) \in C^1(\bar{\Omega})^{++}$ for all $t \in \mathbb{R}$. Globally negative solutions are defined analogously.

It follows from (KR1) that the second coordinate of the trajectory of any nonzero element of $\mathscr{S}$ is either a globally positive or a globally negative solution of $(\mathrm{E}_{\mathbf{b}})$. The next result states that *each* globally positive (or globally negative) solution of $(\mathrm{E}_{\mathbf{b}})$ can be represented in that way.

THEOREM 2.4. $\mathscr{S} = \{(\mathbf{b}\cdot t, u_{\mathbf{b}}(t)) : t \in \mathbb{R}, \mathbf{b} \in \mathbb{A}$ *and* $u_{\mathbf{b}}(\cdot)$ *is a globally positive, globally negative or zero solution of* $(E_{\mathbf{b}})\}$.

Proof. See [14] for the Dirichlet case and [13] for the Robin case. ∎

We write ψ for the mapping ϕ restricted to the subbundle $\mathscr{S}$.

We say that $\lambda \in \mathbb{R}$ belongs to the *upper principal resolvent* of equation (E) if there are constants $K \geqslant 1$ and $\alpha > 0$ such that

$$\|\psi_t(\mathbf{b})(\psi_s(\mathbf{b}))^{-1}\|_2 \leqslant Ke^{(\lambda-\alpha)(t-s)} \qquad \text{for all} \quad \mathbf{b} \in \mathbb{A} \quad \text{and} \quad s \leqslant t. \tag{2.3}$$

We say that $\lambda \in \mathbb{R}$ belongs to the *lower principal resolvent* of (E) if there are constants $0 < K \leqslant 1$ and $\alpha > 0$ such that

$$\|\psi_t(\mathbf{b})(\psi_s(\mathbf{b}))^{-1}\|_2 \geqslant Ke^{(\lambda+\alpha)(t-s)} \qquad \text{for all} \quad \mathbf{b} \in \mathbb{A} \quad \text{and} \quad s \leqslant t. \tag{2.4}$$

The union of the upper and lower principal resolvents is called the *principal resolvent* of (E). By the *principal spectrum* of (E) we mean the complement in $\mathbb{R}$ of the principal resolvent.

We have the following characterization of the principal resolvent.

LEMMA 2.5. (i) *$\lambda \in \mathbb{R}$ is in the upper principal resolvent of (E) if and only if there are constants $K \geqslant 1$ and $\alpha > 0$ such that*

$$\|\psi_{t-s}(\mathbf{b} \cdot s)\|_2 = \frac{\|\phi_t(\mathbf{b})\, v_{\mathbf{b}}\|_2}{\|\phi_s(\mathbf{b})\, v_{\mathbf{b}}\|_2} \leqslant K e^{(\lambda-\alpha)(t-s)} \qquad \text{for all} \quad \mathbf{b} \in \mathbb{A} \quad \text{and} \quad s \leqslant t.$$

(ii) *$\lambda \in \mathbb{R}$ is in the lower principal resolvent of (E) if and only if there are constants $0 < K \leqslant 1$ and $\alpha > 0$ such that*

$$\|\psi_{t-s}(\mathbf{b} \cdot s)\| = \frac{\|\phi_t(\mathbf{b})\, v_{\mathbf{b}}\|_2}{\|\phi_s(\mathbf{b})\, v_{\mathbf{b}}\|_2} \geqslant K e^{(\lambda+\alpha)(t-s)} \qquad \text{for all} \quad \mathbf{b} \in \mathbb{A} \quad \text{and} \quad s \leqslant t.$$

Proof. By the cocycle property $\psi_{t-s}(\mathbf{b} \cdot s) = \psi_t(\mathbf{b})(\psi_s(\mathbf{b}))^{-1}$. Further, as $\mathscr{S}$ has dimension one,

$$\|\psi_t(\mathbf{b})(\psi_s(\mathbf{b}))^{-1}\|_2 = \|\psi_t(\mathbf{b})\|_2 \, \|(\psi_s(\mathbf{b}))^{-1}\|_2$$

$$= \frac{\|\psi_t(\mathbf{b})\|_2}{\|\psi_s(\mathbf{b})\|_2} = \frac{\|\phi_t(\mathbf{b})\, v_{\mathbf{b}}\|_2}{\|\phi_s(\mathbf{b})\, v_{\mathbf{b}}\|_2}.$$

The following result is a consequence of a theorem of Sacker and Sell.

THEOREM 2.6. *The principal spectrum of (E) is a nonempty compact interval.*

Proof. The principal spectrum of (E) equals the dynamical spectrum of the linear skew-product dynamical system ψ on the one-dimensional bundle $\mathscr{S}$ (for the spectral theory of linear skew-product dynamical systems on finite-dimensional vector bundles see the paper [18] by R. J. Sacker and G. R. Sell). Theorem 2 in [18] asserts that the dynamical spectrum of a linear skew-product flow on a bundle with a connected compact base and n-dimensional fibers consists of at most n nonoverlapping nonempty compact intervals. In case $\mathbb{T} = (-\infty, \infty)$ the base space $\mathbb{A}$ is connected as the closure of the connected set $\{\mathbf{a} \cdot t : t \in \mathbb{R}\}$. In case $\mathbb{T} = [0, \infty)$ the space $\mathbb{A}$ is connected as an inverse limit of compact connected spaces. ∎

2.3. *A Characterization of the Principal Spectrum*

As it is (usually) hard to find the subbundle $\mathscr{S}$ for a given equation (E), we proceed now to give a useful characterization of the principal spectrum in terms of (E) only.

We begin by defining some positive solution of (E).

Case $\mathbb{T} = (-\infty, \infty)$. Put $v : (-\infty, \infty) \to L^2(\Omega)$ to be the unique globally positive solution of (E) such that $\|v(0) = 1\|$ (by Theorem 2.4, such a solution exists and is unique).

Case $\mathbb{T} = [0, \infty)$. Define $v : [0, \infty) \to L^2(\Omega)$ as the solution of (E) with the corresponding boundary conditions and the initial condition $v(0) = \mathbf{e}$.

For $0 \leqslant t_0 \leqslant t$ and $u_0 \in L^2(\Omega)$ put $\tilde{\phi}(t; t_0)\, u_0 := u(t - t_0;\ (a_{ij}, a_i, a_0) \cdot t_0, u_0)$. We can interpret $\tilde{\phi}(t; t_0)\, u_0$ as the value at time t of the solution of equation

$$u_t = \sum_{i,\, j=1}^{N} a_{ij}(t, x)\, u_{x_j x_i} + \sum_{i=1}^{N} a_i(t, x)\, u_{x_i} + a_0(t, x)\, u, \qquad t > t_0, \quad x \in \Omega,$$

with appropriate boundary conditions and the initial condition

$$u(t_0, x) = u_0(x).$$

The family $\{\tilde{\phi}(t; s)\}_{0 \leqslant s \leqslant t}$ is a so-called *evolution system* on $L^2(\Omega)$ (see e.g. Chapter 5 in Pazy [16]). In particular, we have as a consequence of (1.1) the *cocycle equality*

$$\tilde{\phi}(t_3; t_1) = \tilde{\phi}(t_3; t_2)\, \tilde{\phi}(t_2; t_1) \qquad 0 \leqslant t_1 \leqslant t_2 \leqslant t_3. \tag{2.5}$$

A simple consequence of Axiom (A5) and the construction of $\mathbb{A}$ is the following result, which we state here for further reference.

LEMMA 2.7. *Let* $\mathbb{T} = [0, \infty)$. *Assume that* $\mathbf{b} \in \mathbb{A}$, $t \in \mathbb{R}$, v_n *is a sequence in* $L^2(\Omega)$ *converging to* v, *and* $t_n \to \infty$ *is a sequence such that* $(a_{ij}, a_i, a_0) \cdot t_n$ *converges to* $\mathbf{b}$ *as* $n \to \infty$. *Then*

$$\|\tilde{\phi}(t_n + t; t)\, v_n - \phi_t(\mathbf{b})\, v\|_2 \to 0 \qquad \text{as} \quad n \to \infty.$$

PROPOSITION 2.8. *Let* $\mathbb{T} = [0, \infty)$. *Assume that* $\mathbf{b} \in \mathbb{A}$ *and* $t_n \to \infty$ *are such that* $(a_{ij}, a_i, a_0) \cdot t_n$ *converges to* $\mathbf{b}$ *as* $n \to \infty$. *Then*

$$\left\| \frac{v(t_n)}{\|v(t_n)\|_2} - v_{\mathbf{b}} \right\|_2 \to 0 \qquad \text{as} \quad n \to \infty.$$

Proof. The first step is to reduce the result to the case of discrete time. Indeed, by passing to a subsequence, if necessary, we can assume that $(a_{ij}, a_i, a_0) \cdot [t_n]$ converges to some $\mathbf{c} \in \mathbb{A}$ and $t_n - [t_n]$ converges to some $s \in [0, 1]$, where $[\cdot]$ stands for the integer part. One has $\mathbf{b} = \mathbf{c} \cdot s$. Suppose that we have already proved that

$$\left\| \frac{v([t_n])}{\|v([t_n])\|_2} - v_{\mathbf{c}} \right\|_2 \to 0 \qquad \text{as} \quad n \to \infty$$

By Lemma 2.7

$$\left\| \tilde{\phi}(t_n; [t_n]) \frac{v([t_n])}{\|v([t_n])\|_2} - \phi_s(\mathbf{c})\, v_{\mathbf{c}} \right\|_2 \to 0 \qquad \text{as} \quad n \to \infty.$$

Since $\phi_s(\mathbf{c})\, v_{\mathbf{c}} \neq 0$, one has that $v(t_n)/\|v(t_n)\|_2$ converges in $L^2(\Omega)$ to $\phi_s(\mathbf{c})\, v_{\mathbf{c}}/\|\phi_s(\mathbf{c})\, v_{\mathbf{c}}\|_2 = v_{\mathbf{b}}$ as $n \to \infty$.

As a consequence of Axiom (A9),

(a) the mapping $\mathbb{A} \ni \mathbf{b} \mapsto \phi_1(\mathbf{b}) \in \mathscr{L}((C^1(\bar{\Omega}))^*, C^1(\bar{\Omega}))$ is continuous,

(b) if $t_n \to \infty$ has the property that $(a_{ij}, a_i, a_0) \cdot t_n \to \mathbf{b}$ as $n \to \infty$ then $\tilde{\phi}(t_n + 1; t_n)$ converges to $\phi_1(\mathbf{b})$ in $\mathscr{L}((C^1(\bar{\Omega}))^*, C^1(\bar{\Omega}))$.

Therefore the family $\{\tilde{\phi}(t+1; t) : t \in [0, \infty)\} \cup \{\phi_1(\mathbf{b}) : \mathbf{b} \in \mathbb{A}\}$ is a compact subset of $\mathscr{L}((C^1(\bar{\Omega}))^*, C^1(\bar{\Omega}))$.

Repeating the reasoning in Section 1 of Mierczyński [14] we prove that $\phi_1(\mathbf{b})$ [resp. $\tilde{\phi}(t+1; t)$] can be written in the form

$$(\phi_1(\mathbf{b})\, u)(x) = \int_{\bar{\Omega}} G(\mathbf{b}; x, \xi)\, u(\xi)\, d\xi, \qquad \mathbf{b} \in \mathbb{A}, \quad u \in L^2(\Omega), \quad x \in \bar{\Omega},$$

$$\left[\text{resp. } (\tilde{\phi}(t+1; t)\, u)(x) = \int_{\bar{\Omega}} \tilde{G}(t; x, \xi)\, u(\xi)\, d\xi, \qquad t \in [0, \infty), \right.$$

$$\left. u \in L^2(\Omega), \quad x \in \bar{\Omega}, \right]$$

where the continuous functions $G\colon \mathbb{A} \times \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ and $\tilde{G}\colon [0, \infty) \times \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ have the following properties:

(G1) The partial derivatives $\partial/\partial x_i$, $\partial/\partial \xi_i$ and $\partial^2/\partial x_i \partial \xi_j$ of G [resp. $\tilde{G}$] are continuous in all variables.

(G2) If $(a_{ij}, a_i, a_0) \cdot t_n$ converges to $\mathbf{b}$ as $n \to \infty$ then $\tilde{G}(t_n; \cdot, \cdot)$ converges to $G(\mathbf{b}; \cdot, \cdot)$ in the Banach space $C(\bar{\Omega} \times \bar{\Omega})$ of continuous real functions on $\bar{\Omega} \times \bar{\Omega}$. The same holds for $\partial/\partial x_i$, $\partial/\partial \xi_i$ and $\partial^2/\partial x_i \partial \xi_j$.

(G3) $G(\mathbf{b};\,\cdot,\xi)\in C^1_{\mathscr{B}}(\bar{\Omega})^{++}$ for $\mathbf{b}\in\mathbb{A}$ and $\xi\in\Omega$ [resp. $\tilde{G}(t;\cdot,\xi)\in C^1_{\mathscr{B}}(\bar{\Omega})^{++}$ for $t\geqslant 0$ and $\xi\in\Omega$].

(G4) $G(\mathbf{b};x,\cdot)\in C^1_{\mathscr{B}}(\bar{\Omega})^{++}$ for $\mathbf{b}\in\mathbb{A}$ and $x\in\Omega$ [resp. $\tilde{G}(t;x,\cdot)\in C^1_{\mathscr{B}}(\bar{\Omega})^{++}$ for $t\geqslant 0$ and $x\in\Omega$].

(G5) $(\partial^2 G/\partial\nu_{x_i}\partial\nu_{\xi_j})(\mathbf{b};x,\xi)<0$ for $\mathbf{b}\in\mathbb{A}$ and $(x,\xi)\in\partial\Omega\times\partial\Omega$ [resp. $(\partial^2\tilde{G}/\partial\nu_{x_i}\partial\nu_{\xi_j})(t;x,\xi)<0$ for $t\geqslant 0$ and $(x,\xi)\in\partial\Omega\times\partial\Omega$].

Modifying appropriately the proof of Proposition 2.9 in [14] we obtain from (G1)–(G5) that there is a constant $\kappa>1$ such that

(i) for each $\mathbf{b}\in\mathbb{A}$ and $u\in L^2(\Omega)^+\backslash\{0\}$ we can find $\alpha=\alpha(\mathbf{b},u)>0$ such that

$$\alpha(\mathbf{b},u)\,\mathbf{e}\leqslant_{\mathbf{e}}\phi_1(\mathbf{b})\,u\leqslant_{\mathbf{e}}\kappa\alpha(\mathbf{b},u)\,\mathbf{e}, \tag{2.6}$$

and

(ii) for each $t\in[0,\infty)$ and $u\in L^2(\Omega)^+\backslash\{0\}$ we can find $\tilde{\alpha}=\tilde{\alpha}(t,u)>0$ such that

$$\tilde{\alpha}(t,u)\,\mathbf{e}\leqslant_{\mathbf{e}}\tilde{\phi}(t+1;t)\,u\leqslant_{\mathbf{e}}\kappa\tilde{\alpha}(t,u)\,\mathbf{e}. \tag{2.7}$$

Put

$$\chi_n(\mathbf{b})\,u:=\frac{\phi_n(\mathbf{b})\,u}{\|\phi_n(\mathbf{b})\,u\|_2}\qquad\text{and}\qquad\tilde{\chi}_n(t)\,u:=\frac{\tilde{\phi}(t+n;t)\,u}{\|\tilde{\phi}(t+n;t)\,u\|_2}.$$

We write for simplicity $\chi(\mathbf{b})$ instead of $\chi_1(\mathbf{b})$ and $\tilde{\chi}(t)$ instead of $\tilde{\chi}_1(t)$. The cocycle equalities (2.1) and (2.5) imply that

$$\chi_n(\mathbf{b})=\chi(\mathbf{b}\cdot(n-1))\cdots\chi(\mathbf{b})\qquad\text{and}\qquad\tilde{\chi}_n(t)=\tilde{\chi}(t+n-1))\cdots\tilde{\chi}(t)$$

for all $n\in\mathbb{N}$, $\mathbf{b}\in\mathbb{A}$ and $t\in[0,\infty)$.

By Axiom (A10), $\chi(\mathbf{b})$ and $\tilde{\chi}(t)$ carry the set $\mathbb{S}:=\{u\in L^2(\Omega)^+:\|u\|_2=1\}$ into itself. It is a simple consequence of properties (2.6) and (2.7) that $\chi(\mathbf{b})\,\mathbb{S}\subset\{u\in L^2(\Omega)^{++}_{\mathbf{e}}:\|u\|_2=1\}$ and $\tilde{\chi}(t)\,\mathbb{S}\subset\{u\in L^2(\Omega)^{++}_{\mathbf{e}}:\|u\|_2=1\}$.

Another consequence of (2.6) and (2.7) is:

$$d(\chi(\mathbf{b})\,u,\mathbf{e})\leqslant\log\kappa\qquad\text{and}\qquad d(\tilde{\chi}(t)\,u,\mathbf{e})\leqslant\log\kappa$$

for all $\mathbf{b}\in\mathbb{A}$, $t\in[0,\infty)$ and $u\in\mathbb{S}$, where $d(\cdot,\,\cdot)$ denotes the Hilbert projective metric (see e.g. Nussbaum [15]). Therefore, the projective diameters of the images $\chi(\mathbf{b})\,\mathbb{S}$ and $\tilde{\chi}(t)\,\mathbb{S}$ are bounded by $2\log\kappa$, uniformly in $\mathbb{A}$ and

$t \in [0, \infty)$. A celebrated result due originally to Garrett Birkhoff (see Theorem 2.3 in [15]) gives that

$$d(\chi(\mathbf{b})\, u, \chi(\mathbf{b})\, v) \leqslant \gamma d(u, v) \qquad \text{and} \qquad d(\tilde{\chi}(t)\, u, \tilde{\chi}(t)\, v) \leqslant \gamma d(u, v)$$

for all $\mathbf{b} \in \mathbb{A}$, $t \in [0, \infty)$, $u, v \in \mathbb{S} \cap L^2(\Omega)_{\mathbf{e}}^{++}$, where $\gamma := \tanh(\kappa/2) \in (0, 1)$.

Let $k_n \to \infty$ be a sequence of positive integers such that $(a_{ij}, a_i, a_0) \cdot k_n \to \mathbf{b} \in \mathbb{A}$ as $n \to \infty$. From the normality of $L^2(\Omega)^+$ it follows that the set $\mathbb{S} \cap L^2(\Omega)_{\mathbf{e}}^{++}$ with the projective distance is a complete metric space (see Theorem 1.1 in [15]), hence there exists $v_{\lim} \in \mathbb{S} \cap L^2(\Omega)_{\mathbf{e}}^{++}$ such that $d(\tilde{\chi}_{k_n}(0)\, v(0), v_{\lim}) \to 0$ as $n \to \infty$. By the construction of the space $L^2(\Omega)_{\mathbf{e}}$ it follows that $\|\tilde{\chi}_{k_n}(0)\, v(0) - v_{\lim}\|_2 \to 0$ as $n \to \infty$.

We claim that $v_{\lim} = v_{\mathbf{b}}$. Suppose not. Put $d := d(v_{\lim}, v_{\mathbf{b}}) > 0$, and $l \in \mathbb{N}$ so large that $2(\log \kappa)\, \gamma^l < d$ (this is possible as $\gamma < 1$). There exists $v'_{\lim} \in \mathbb{S} \cap L^2(\Omega)_{\mathbf{e}}^{++}$ such that $\|\tilde{\chi}_{k_n - l}(0)\, v(0) - v'_{\lim}\|_2 \to 0$. As $(a_{ij}, a_i, a_0) \cdot (k_n - l)$ converges to $\mathbf{b} \cdot (-l)$, we get from Lemma 2.7 that $\tilde{\phi}(k_n; k_n - l)\, \tilde{\chi}_{k_n - l}(0)\, v(0)$ converges (in $L^2(\Omega)$) to $\phi_l(\mathbf{b} \cdot (-l))\, v'_{\lim}$. As the latter is nonzero, simple calculation yields $\tilde{\chi}_l(k_n - l)\, \tilde{\chi}_{k_n - l}(0)\, v(0) = \tilde{\chi}_{k_n}(0)\, v(0)$ converges (in $L^2(\Omega)$) to $\chi_l(\mathbf{b} \cdot (-l))\, v'_{\lim}$. Consequently, $v_{\lim} = \chi^{(l)}(\mathbf{b} \cdot (-l))\, v'_{\lim}$. But $d(v'_{\lim}, v_{\mathbf{b} \cdot (-l)}) \leqslant 2 \log \kappa$, hence $d(v_{\lim}, v_{\mathbf{b}}) \leqslant 2(\log \kappa)\, \gamma^l < d$, a contradiction. ∎

In the classical case (see Subsection 1.1(I)) the above proposition can be proved in a simpler way. We sketch here briefly the idea (due to an anonymous referee). Consider

$$u_t = \Delta u + a_0(t, x)\, u, \qquad t \in [0, \infty), \quad x \in \Omega,$$

with the homogeneous Dirichlet or Robin boundary conditions, where $a_0 \colon [0, \infty) \times \bar{\Omega} \to \mathbb{R}$ is a bounded continuous function, uniformly Hölder in x with exponent α, $0 < \alpha \leqslant 1$, and uniformly Hölder in t with exponent $\alpha/2$. We extend a_0 to $(-\infty, \infty) \times \bar{\Omega}$ by putting:

$$\bar{a}_0(t, x) = \begin{cases} a_0(t, x) & \text{for} \quad t \geqslant 0 \\ (1 + \tanh t) \cdot a_0(0, x) & \text{for} \quad t < 0 \end{cases}$$

Further, we write $a_{ij} = \bar{a}_{ij} \equiv \delta_{ij} 1$, $i, j = 1, ..., N$, where δ_{ij} stands for the Kronecker delta, $a_i = \bar{a}_i \equiv 0$ for $i = 1, ..., N$. A standard application of the Ascoli–Arzelà theorem yields that the set of all t-translates, $t \in (-\infty, \infty)$, of $(\bar{a}_{ij}, \bar{a}_i, \bar{a}_0)$ has compact closure in the Fréchet space $C((-\infty, \infty) \times \bar{\Omega}, \mathbb{R}^{N^2 + N + 1})$ with the topology of uniform C^0 convergence on compact sets. Put $\bar{\mathbb{A}} := \mathscr{H}((\bar{a}_{ij}, \bar{a}_i, \bar{a}_0))$. Observe that for $\mathbf{b} \in \mathbb{A}$ and a sequence $t_n \to \infty$ the sentence "$(a_{ij}, a_i, a_0) \cdot t_n \to \mathbf{b}$" (understood as in Subsection 2.1) is equivalent to saying "$(\bar{a}_{ij}, \bar{a}_i, \bar{a}_0) \cdot t_n$ in the Fréchet space $C((-\infty, \infty) \times \bar{\Omega}$,

$\mathbb{R}^{N^2+N+1}$)." We can therefore identify $\mathbb{A}$ with a compact subset of $\bar{\mathbb{A}}$, invariant under t-translation. Moreover, for any sequence $t_n \to -\infty$ the sequence $(\bar{a}_{ij}, \bar{a}_i, \bar{a}_0) \cdot t_n$ converges in $C((-\infty, \infty) \times \bar{\Omega}, \mathbb{R}^{N^2+N+1})$ to $(\bar{a}_{ij}, \bar{a}_i, 0)$ (that is, to the Laplace equation). In the language of dynamical systems we can say that $\alpha((\bar{a}_{ij}, \bar{a}_i, \bar{a}_0)) = \{\text{Laplace equation}\}$ and $\omega((\bar{a}_{ij}, \bar{a}_i, \bar{a}_0)) = \mathbb{A}$, where $\alpha(\cdot)$ and $\omega(\cdot)$ stand for alpha- and omega-limit sets (see e.g. Hale [8]) with respect to the translation dynamical system on $\bar{\mathbb{A}}$.

We define a linear skew-product semidynamical system $\bar{\Phi} = \{\bar{\Phi}_t\}_{t \geq 0}$ on the Banach bundle $\bar{\mathscr{B}} = \bar{\mathbb{A}} \times L^2(\Omega)$ by

$$\bar{\Phi}_t((\bar{b}_{ij}, \bar{b}_i, \bar{b}_0), u) := ((\bar{b}_{ij}, \bar{b}_i, \bar{b}_0) \cdot t, \phi_t((\bar{b}_{ij}, \bar{b}_i, \bar{b}_0))\, u)$$

for $t \geq 0$, $(\bar{b}_{ij}, \bar{b}_i, \bar{b}_0) \in \bar{\mathbb{A}}$ and $u \in L^2(\Omega)$, where $\phi_t((\bar{b}_{ij}, \bar{b}_i, \bar{b}_0))\, u_0$ is a solution of

$$u_t = \Delta u + \bar{b}_0(t, x)\, u, \qquad t > 0, \quad x \in \Omega$$

with appropriate boundary conditions and initial condition

$$u(0, x) = u_0(x).$$

Theorem 2.3 applied to $\bar{\Phi}$ provides the existence of two invariant subbundles, $\bar{\mathscr{S}}$ and $\bar{\mathscr{T}}$, satisfying properties (KR1)–(KR3). The subbundle $\bar{\mathscr{S}}$ (resp. $\bar{\mathscr{T}}$) restricted to $\mathbb{A}$ equals $\mathscr{S}$ (resp. $\mathscr{T}$). Since, by (KR2), $((\bar{a}_{ij}, \bar{a}_i, \bar{a}_0), v(0))$ does not belong to $\bar{\mathscr{T}}$, exponential separation (KR3) for $\bar{\mathscr{S}}$ and $\bar{\mathscr{T}}$ yields the convergence of $((\bar{a}_{ij}, \bar{a}_i, \bar{a}_0) \cdot t_n, v(t_n)/\|v(t_n)\|_2)$ to $(\mathbf{b}, v_{\mathbf{b}})$ in $\bar{\mathbb{A}} \times L^2(\Omega)$.

A consequence of Proposition 2.8 and the construction of $\mathbb{A}$ is the following.

THEOREM 2.9. *Assume $\mathbb{T} = [0, \infty)$. Then for each $t \geq 0$ the set*

$$\bigcup_{s \geq t} \frac{v(s)}{\|v(s)\|_2}$$

has compact closure in $L^2(\Omega)$. Furthermore,

$$\bigcap_{t \geq 0} \operatorname{cl}\left(\bigcup_{s \geq t} \frac{v(s)}{\|v(s)\|_2}\right) = \{u \in \mathscr{S}_{\mathbf{b}} \cap L^2(\Omega)^+ : \mathbf{b} \in \mathbb{A},\ \|u\|_2 = 1\},$$

where cl *denotes the closure in $L^2(\Omega)$.*

We present now the promised characterization of the principal spectrum.

THEOREM 2.10. *Assume* $\mathbb{T} = (-\infty, \infty)$. *Then*

(i) $\lambda \in \mathbb{R}$ *is in the upper principal resolvent of* (E) *if and only if there are constants* $K \geqslant 1$ *and* $\alpha > 0$ *such that*

$$\frac{\|v(t)\|_2}{\|v(s)\|_2} \leqslant K e^{(\lambda - \alpha)(t - s)} \tag{2.8}$$

for all $s \leqslant t$.

(ii) $\lambda \in \mathbb{R}$ *is in the lower principal resolvent of* (E) *if and only if there are constants* $0 < K \leqslant 1$ *and* $\alpha > 0$ *such that*

$$\frac{\|v(t)\|_2}{\|v(s)\|_2} \geqslant K e^{(\lambda + \alpha)(t - s)} \tag{2.9}$$

for all $s \leqslant t$.

Assume $\mathbb{T} = [0, \infty)$. *Then*

(iii) $\lambda \in \mathbb{R}$ *is in the upper principal resolvent of* (E) *if and only if there are constants* $K \geqslant 1$ *and* $\alpha > 0$ *with the property that for each* $r \geqslant 0$ *there is* $T \geqslant 0$ *such that*

$$\frac{\|v(t)\|_2}{\|v(s)\|_2} \leqslant K e^{(\lambda - \alpha)(t - s)} \tag{2.8}$$

for all $T \leqslant s \leqslant t$ *with* $t - s = r$.

(iv) $\lambda \in \mathbb{R}$ *is in the lower principal resolvent of* (E) *if and only if there are constants* $0 < K \leqslant 1$ *and* $\alpha > 0$ *with the property that for each* $r \geqslant 0$ *there is* $T \geqslant 0$ *such that*

$$\frac{\|v(t)\|_2}{\|v(s)\|_2} \geqslant K e^{(\lambda + \alpha)(t - s)} \tag{2.9}$$

for all $T \leqslant s \leqslant t$ *with* $t - s = r$.

Proof. In case $\mathbb{T} = (-\infty, \infty)$, as $\{\mathbf{a} \cdot t : t \in \mathbb{R}\}$ is dense in $\mathbb{A}$, the result follows by Sacker and Sell ([18]).

So, let $\mathbb{T} = [0, \infty)$. We prove only part (iii), the proof of (iv) being similar. Assume that λ belongs to the upper principal resolvent, that is, there are $K' \geqslant 1$ and $\alpha > 0$ such that

$$\|\phi_{t-s}(\mathbf{b} \cdot s)\, v_{\mathbf{b} \cdot s}\|_2 < K' e^{(\lambda - \alpha)(t - s)}$$

for all $\mathbf{b} \in \mathbb{A}$ and $s \leqslant t$ (see Lemma 2.5). Fix $r = t - s$. If $r = 0$ there is nothing to prove. So assume $r > 0$. By the construction of $\mathbb{A}$, Axiom (A5) and Proposition 2.8 it follows that each of the sets $\bigcup_{\tau \geqslant \theta} \tilde{\phi}(\tau + r; \tau)\, v(\tau)/\|v(\tau)\|_2$, $\theta > 0$, has compact closure in $L^2(\Omega)$, and that the intersection of their closures equals $\{\phi_r(\mathbf{b})\, v_{\mathbf{b}} : \mathbf{b} \in \mathbb{A}\}$. Consequently, there is $T \geqslant 0$ such that $\|\tilde{\phi}(s + r; s)\, v(s)\|_2/\|v(s)\|_2 < K' e^{(\lambda - \alpha) r}$ for all $s \geqslant T$. But $\tilde{\phi}(s + r; s)\, v(s) = v(s + r) = v(t)$.

To prove the converse implication, it suffices to notice that $\phi_{t-s}(\mathbf{b} \cdot s)\, v_{\mathbf{b} \cdot s}$ can be approximated by $\tilde{\phi}(t_n + t - s;\ t_n - s)\, v(t_n - s)/\|v(t_k - s)\|_2$ for some $t_n \to \infty$ (see Lemma 2.7 and Proposition 2.8). ∎

See Theorem 2.14 for an improvement on the above result.

2.4. *Another Characterization of the Principal Spectrum*

Denote the principal spectrum of (E) by $[\underline{\lambda}, \bar{\lambda}]$.

Proposition 2.11. (i) *Assume* $\mathbb{T} = (-\infty, \infty)$. *Then*

$$\underline{\lambda} = \liminf_{t - s \to \infty} \frac{\log \|v(t)\|_2 - \log \|v(s)\|_2}{t - s} \leqslant \limsup_{t - s \to \infty} \frac{\log \|v(t)\|_2 - \log \|v(s)\|_2}{t - s} = \bar{\lambda}.$$

(ii) *Assume* $\mathbb{T} = [0, \infty)$. *Then*

$$\underline{\lambda} = \liminf_{\substack{t - s \to \infty \\ s \to \infty}} \frac{\log \|v(t)\|_2 - \log \|v(s)\|_2}{t - s} \leqslant \limsup_{\substack{t - s \to \infty \\ s \to \infty}} \frac{\log \|v(t)\|_2 - \log \|v(s)\|_2}{t - s} = \bar{\lambda}.$$

Proof. Assume $\mathbb{T} = (-\infty, \infty)$. The fact that

$$\underline{\lambda} \leqslant \liminf_{t - s \to \infty} \frac{\log \|v(t)\|_2 - \log \|v(s)\|_2}{t - s} \leqslant \limsup_{t - s \to \infty} \frac{\log \|v(t)\|_2 - \log \|v(s)\|_2}{t - s} \leqslant \bar{\lambda}.$$

is an easy consequence of Theorem 2.10(i)–(ii).

We now prove the equality for $\bar{\lambda}$, the proof for $\underline{\lambda}$ being similar. By Theorem 2.3 in [11] there is an invariant probability ergodic measure μ on $\mathbb{A}$ such that the set of all those $\mathbf{b} \in \mathbb{A}$ for which the equality

$$\lim_{t \to \infty} \frac{\log \|\psi_t(\mathbf{b})\|_2}{t} = \bar{\lambda}$$

holds, has μ-measure 1. Fix some $\mathbf{c} \in \mathbb{A}$ having this property, and let s_n be a sequence such that $\mathbf{a} \cdot s_n \to \mathbf{c}$ as $n \to \infty$. For each $\varepsilon > 0$ take T so large that $\log \|\psi_\tau(\mathbf{b})\|_2/\tau \geqslant \bar{\lambda} - (\varepsilon/2)$ for all $\tau \geqslant T$. Further, for a fixed $t \geqslant T$ one can find $k = k(t) \in \mathbb{N}$ such that $\log \|\psi_t(\mathbf{a} \cdot s_k)\|_2/s_k \geqslant \bar{\lambda} - \varepsilon$. As $\log \|\psi_t(\mathbf{a} \cdot s_k)\|_2 = \log \|v(t + s_k)\|_2 - \log \|v(s_k)\|_2$, we obtain the desired result.

In case $\mathbb{T} = [0, \infty)$ the result follows from Theorem 10.1 in [11]. ∎

A simple consequence of the above result is the following.

THEOREM 2.12. (i) *Assume* $\mathbb{T} = (-\infty, \infty)$. *Then for each* $\lambda \in [\underline{\lambda}, \bar{\lambda}]$ *there are sequences* $s_n < t_n$, $t_n - s_n \to \infty$ *as* $n \to \infty$, *such that*

$$\lim_{n \to \infty} \frac{\log \|v(t_n)\|_2 - \log \|v(s_n)\|_2}{t_n - s_n} = \lambda.$$

(ii) *Assume* $\mathbb{T} = [0, \infty)$. *Then for each* $\lambda \in [\underline{\lambda}, \bar{\lambda}]$ *there are sequences* $s_n < t_n$, $s_n \to \infty$ *and* $t_n - s_n \to \infty$ *as* $n \to \infty$, *such that*

$$\lim_{n \to \infty} \frac{\log \|v(t_n)\|_2 - \log \|v(s_n)\|_2}{t_n - s_n} = \lambda.$$

Proof. Write $\lambda \in [\underline{\lambda}, \bar{\lambda}]$ as $\lambda = \alpha \underline{\lambda} + (1 - \alpha) \bar{\lambda}$ with $\alpha \in [0, 1]$. Let $s'_n < t'_n$ and $s''_n < t''_n$ be sequences (whose existence is guaranteed by Proposition 2.11) such that

$$\lim_{n \to \infty} \frac{\log \|v(t'_n)\|_2 - \log \|v(s'_n)\|_2}{t'_n - s'_n} = \lambda$$

and

$$\lim_{n \to \infty} \frac{\log \|v(t''_n)\|_2 - \log \|v(s''_n)\|_2}{t''_n - s''_n} = \bar{\lambda}.$$

For each $n \in \mathbb{N}$ there is $\beta = \beta(n) \in [0, 1]$ such that

$$\frac{\log \|v(t_n)\|_2 - \log \|v(s_n)\|_2}{t_n - s_n}$$
$$= \alpha \frac{\log \|v(t'_n) - \log \|v(s'_n)\|_2}{t'_n - s'_n} + (1-\alpha) \frac{\log \|v(t''_n)\|_2 - \log \|v(s''_n)\|_2}{t''_n - s''_n},$$

where $s_n = \beta s'_n + (1-\beta)\, s''_n$ and $t_n = \beta t'_n + (1-\beta)\, t''_n$. It is straightforward that $t_n - s_n \to \infty$, and, moreover, $s_n \to \infty$ in case $\mathbb{T} = [0, \infty)$.

Standard results in the theory of Lyapunov exponents give the following.

THEOREM 2.13. *Assume* $\mathbb{T} = [0, \infty)$. *Then*

$$\lambda\mathrm{UB} \leqslant \liminf_{t \to \infty} \frac{\log \|v(t)\|_2}{t} \leqslant \limsup_{t \to \infty} \frac{\log \|v(t)\|_2}{t} \leqslant \bar{\lambda}.$$

Proof. In the theory of Lyapunov exponents the expression $\limsup_{t \to \infty} (\log \|v(t)\|_2)/t$ is called the *upper Lyapunov exponent*, and $\bar{\lambda}$ is the *upper singular exponent*. By the results contained in 7.4 (Chapter 3) of the book [4] of B. F. Bylov, R. È. Vinograd, D. M. Grobman and V. V. Nemytskiĭ, the former does not exceed the latter. ∎

It should be emphasized here that the above inequalities can be proper. We are now in a position to improve Theorem 2.10 a little.

THEOREM 2.14. *Assume* $\mathbb{T} = [0, \infty)$. *Then*

(i) $\lambda \in \mathbb{R}$ *is in the upper principal resolvent of* (E) *if and only if there are constants* $K \geqslant 1$ *and* $\alpha > 0$ *such that*

$$\frac{\|v(t)\|_2}{\|v(s)\|_2} \leqslant K e^{(\lambda - \alpha)(t - s)} \tag{2.10}$$

for all $0 \leqslant s \leqslant t$.

(ii) $\lambda \in \mathbb{R}$ *is in the lower principal resolvent of* (E) *if and only if there are constants* $0 < K \leqslant 1$ *and* $\alpha > 0$ *such that*

$$\frac{\|v(t)\|_2}{\|v(s)\|_2} \geqslant K e^{(\lambda + \alpha)(t - s)} \tag{2.11}$$

for all $0 \leqslant s \leqslant t$.

Proof. It remains to prove only that $\lambda \in \mathbb{R}$ belonging to the upper principal resolvent implies (2.10). From Lemma 2.7 and Proposition 2.8 we derive that the set $\{\|v(t)\|_2/\|v(s)\|_2 : 0 \leqslant s \leqslant t, t-s \leqslant 1\}$ is bounded. It suffices therefore to check (2.10) for $t-s>1$. Suppose to the contrary that for each $n \in \mathbb{N}$ one can find $0 \leqslant s_n < t_n - 1$ such that

$$\frac{\|v(t_n)\|_2}{\|v(s_n)\|_2} > n e^{(\lambda - 1/n)}(t_n - s_n).$$

We have $t_n - s_n \to \infty$, since otherwise we would have an unbounded subsequence $\|v(t_{n_k})\|_2/\|v(t_{n_k})\|_2$ with $t_{n_k} - s_{n_k}$ bounded and bounded away from zero, which contradicts the fact that the family $\{\phi(t_{n_k}; s_{n_k}) : k \in \mathbb{N}\}$ is a precompact subset of $\mathscr{L}(L^2(\Omega))$ (Axioms (A5)–(A7)).

Therefore we have

$$\limsup_{n \to \infty} \frac{\log \|v(t_n)\|_2 - \log \|v(s_n)\|_2}{t_n - s_n} \geqslant \lambda,$$

which contradicts Proposition 2.11 (in case of s_n unbounded) or Theorem 2.13 (in case of s_n bounded).

The proof of part (ii) is similar. ∎

Recall that the dynamical system on $\mathbb{A}$ is called *uniquely ergodic* if there is precisely one invariant measure on $\mathbb{A}$. A well-known sufficient condition for unique ergodicity is almost periodicity.

THEOREM 2.15. *Assume that $\mathbb{A}$ is uniquely ergodic. Then the principal spectrum of* (E) *consists of one point.*

Proof. See Theorem 8.1 and Remark 8.2 in Johnson, Palmer and Sell [11]. ∎

3. MONOTONE DEPENDENCE OF THE PRINCIPAL SPECTRUM ON ZERO ORDER TERMS

Before stating and proving the main results we present the following useful results.

LEMMA 3.1. *Assume $\mathbb{T} = (-\infty, \infty)$. Then there exists a constant $L \geqslant 1$ such that for each $(\mathbf{b}, u) \in \mathscr{S}$ with $u \in C(\bar{\Omega})_{\mathbf{e}}^{++}$ there is $\beta = \beta(\mathbf{b}, u) > 0$ such that $\beta \mathbf{e} \leqslant_{\mathbf{e}} u \leqslant_{\mathbf{e}} L\beta \mathbf{e}$.*

Proof. Take $\mathbf{b} \in \mathbb{A}$ and $u \in C(\bar{\Omega})^{++}_{\mathbf{e}}$ such that $\|u\|_{\mathbf{e}} = 1$ and $(\mathbf{b}, u) \in \mathcal{S}$. Let $\alpha > 0$ be such that $\alpha \mathbf{e} \leqslant_{\mathbf{e}} u$. By continuity and the fact that $C(\bar{\Omega})^{++}_{\mathbf{e}}$ is open, there is a neighborhood U of $\mathbf{b}$ in $\mathbb{A}$ such that $\alpha \mathbf{e} \leqslant_{\mathbf{e}} w$ for all $(\mathbf{c}, w) \in \mathcal{S}$ with $\mathbf{c} \in U$ and $\|w\|_{\mathbf{e}} = 1$. Consequently, the function $\mathbb{A} \ni \mathbf{b} \mapsto \alpha_1(\mathbf{b}) := \sup\{\alpha > 0 : \alpha \mathbf{e} \leqslant_{\mathbf{e}} u \text{ for } u \in C(\bar{\Omega})^{++}_{\mathbf{e}} \text{ with } (\mathbf{b}, u) \in \mathcal{S} \text{ and } \|u\|_{\mathbf{e}} = 1\}$ is lower semicontinuous. Similarly, the function $\mathbb{A} \ni \mathbf{b} \mapsto \alpha_2(\mathbf{b}) := \inf\{\alpha > 0 : u \leqslant_{\mathbf{e}} \alpha \mathbf{e} \text{ for } u \in C(\bar{\Omega})^{++}_{\mathbf{e}} \text{ with } (\mathbf{b}, u) \in \mathcal{S} \text{ and } \|u\|_{\mathbf{e}} = 1\}$ is upper semicontinuous. Therefore the function $\mathbf{b} \mapsto \alpha_2(\mathbf{b})/\alpha_1(\mathbf{b})$ is upper semicontinuous, hence it attains its largest value, L, on the compact space $\mathbb{A}$. ▮

LEMMA 3.2. *Assume $\mathbb{T} = [0, \infty)$. Then there exists a constant $L \geqslant 1$ such that for each $t \geqslant 1$ there is $\beta = \beta(t) > 0$ such that $\beta \mathbf{e} \leqslant_{\mathbf{e}} v(t) \leqslant_{\mathbf{e}} L\beta \mathbf{e}$.*

Proof. See (2.7) in the proof of Proposition 2.8. ▮

We consider two equations

$$u_t = \sum_{i,j=1}^{N} a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(t, x) \frac{\partial u}{\partial x_i} + a_{0,1}(t, x)\, u, \qquad t \in \mathbb{T}, \quad x \in \Omega, \tag{E$_1$}$$

and

$$u_t = \sum_{i,j=1}^{N} a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(t, x) \frac{\partial u}{\partial x_i} + a_{0,2}(t, x)\, u, \qquad t \in \mathbb{T}, \quad x \in \Omega, \tag{E$_2$}$$

with the same boundary conditions, where we assume that $a_{0,1}(t, x) \leqslant a_{0,2}(t, x)$ for a.e. $(t, x) \in \mathbb{T} \times \Omega$.

Denote the principal spectrum for equation (E$_i$) by $[\underline{\lambda}_i, \bar{\lambda}_i]$.

THEOREM 3.3. *$\underline{\lambda}_1 \leqslant \underline{\lambda}_2$ and $\bar{\lambda}_1 \leqslant \bar{\lambda}_2$.*

Proof. In case $\mathbb{T} = (-\infty, \infty)$ denote by $v_i : (-\infty, \infty) \to L^2(\Omega)$ the unique global solution of (E$_i$) such that $\|v_i(0)\|_2 = 1$, and in case $\mathbb{T} = [0, \infty)$ denote by $v_i : [0, \infty) \to L^2(\Omega)$ the (unique) solution of (E$_i$) with the initial condition $v_i(0) = \mathbf{e}$. Fix $s < t$ (and, moreover, $s \geqslant 1$ in case $\mathbb{T} = [0, \infty)$). Since $v_1(s), v_2(s) \in C(\bar{\Omega})^{++}_{\mathbf{e}}$, the infimum δ of those $\gamma > 0$ such that $v_1(s) \leqslant_{\mathbf{e}} \gamma v_2(s)$ is positive. By Axiom (A11), $v_1(\tau) \leqslant_{\mathbf{e}} \delta v_2(\tau)$ for $\tau \in [s, t]$. As the norm $\|\cdot\|_2$ is monotone, we have $\|v_1(\tau)\|_2 \leqslant \delta\, \|v_2(\tau)\|_2$ for $\tau \in [s, t]$. Lemma 3.1 in case $\mathbb{T} = (-\infty, \infty)$ or Lemma 3.2 in case $\mathbb{T} = [0, \infty)$ provides the existence of constants $L_1, L_2 \geqslant 1$ such that for each $\tau \in (-\infty, \infty)$ or each $\tau \in [1, \infty)$ there are $\beta_1(\tau), \beta_2(\tau) > 0$ with

$$\beta_1(\tau)\, \mathbf{e} \leqslant_{\mathbf{e}} v_1(\tau) \leqslant_{\mathbf{e}} L_1 \beta_1(\tau)\, \mathbf{e} \qquad \text{and} \qquad \beta_2(\tau)\, \mathbf{e} \leqslant_{\mathbf{e}} v_2(\tau) \leqslant_{\mathbf{e}} L_2 \beta_2(\tau)\, \mathbf{e}.$$

From this it follows by simple calculation that for each $\tau \in (-\infty, \infty)$ or each $\tau \in [1, \infty)$ there is $\beta_3(\tau) > 0$ such that $\beta_3(\tau)\, v_2(\tau) \leqslant_{\mathbf{e}} v_1(\tau) \leqslant_{\mathbf{e}} L_3 \beta_3(\tau)\, v_2(\tau)$, where $L_3 := L_1 L_2$ is independent of τ. As $v_1(s) \leqslant_{\mathbf{e}} \delta v_2(s)$ and δ is the smallest positive number with this property, one has $\delta \leqslant L_3 \beta_3(s)$. This implies that $(\delta/L_3)\, v_2(s) \leqslant_{\mathbf{e}} \beta_3(s)\, v_2(s) \leqslant_{\mathbf{e}} v_1(s)$.

Assume that λ is in the upper resolvent for (E_2). We write

$$\frac{\|v_1(t)\|_2}{\|v_1(s)\|_2} \leqslant \delta\, \|v_2(t)\|_2 \frac{\|L_3\|_2}{\delta\, \|v_2(s)\|_2}$$

$$= L_3 \frac{\|v_2(t)\|_2}{\|v_2(s)\|_2} \leqslant L_3 K e^{(\lambda - \alpha)(t - s)}. \tag{3.1}$$

As L_3 is independent of $s < t$, this means that λ is in the upper resolvent for (E_1), too. As a consequence, $\bar{\lambda}_1 \leqslant \bar{\lambda}_2$.

The proof of the inequality $\underline{\lambda}_1 \leqslant \underline{\lambda}_2$ goes along much the same lines. ∎

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